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Geometry of quaternionic Kähler connections with torsion☆

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Abstract

The target space of a (4, 0) supersymmetric two-dimensional sigma model with Wess–Zumino term has a connection with totally skew-symmetric torsion and holonomy contained in $\mathrm{SP}(n)\cdot\mathrm{SP}(1)$, QKT connection. We study the geometry of QKT connections. We find conditions to the existence of a QKT connection and prove that if it exists it is unique. We show that QKT geometry persist in a conformal class of metrics which allows us to obtain a lot of (compact) examples of QKT manifolds. We present a (local) description of four-dimensional homogeneous QKT structures relying on the known result of naturally reductive homogeneous Riemannian manifolds. We consider Einstein-like QKT manifold and find closed relations with Einstein–Weyl geometry in dimension 4.

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1. Introduction and statement of the results

An almost hypercomplex structure on a $4n$-dimensional manifold M is a triple $H = (J_\alpha), \alpha = 1, 2, 3$, of almost complex structures $J_\alpha : TM \to TM$ satisfying the quaternionic identities $J_\alpha^2 = -\mathrm{id}$ and $J_1J_2 = -J_2J_1 = J_3$. When each J_α is a complex structure, H is said to be a hypercomplex structure on M.

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An almost quaternionic structure on M is a rank-3 subbundle $Q \subset \mathrm{End}(TM)$ which is locally spanned by almost hypercomplex structure $H = (J_\alpha)$; such a locally defined triple H will be called an admissible basis of Q. A linear connection ∇ on TM is called quaternionic connection if ∇ preserves Q, i.e. $\nabla_X \sigma \in \Gamma(Q)$ for all vector fields X and smooth sections $\sigma \in \Gamma(Q)$. An almost quaternionic structure is said to be a quaternionic if there is a torsion-free quaternionic connection. A Q-Hermitian metric is a Riemannian metric which is Hermitian with respect to each almost complex structure in Q. An almost quaternionic (resp. quaternionic) manifold with Q-Hermitian metric is called an almost quaternionic Hermitian (resp. quaternionic Hermitian) manifold.

For $n = 1$, an almost quaternionic structure is the same as an oriented conformal structure and it turns out to be always quaternionic. When $n \geq 2$, the existence of torsion-free quaternionic connection is a strong condition which is equivalent to the 1-integrability of the associated $\mathrm{GL}(n, H)\mathrm{SP}(1)$ structure [10,33,43]. If the Levi-Civita connection of a quaternionic Hermitian manifold (M, g, Q) is a quaternionic connection then (M, g, Q) is called quaternionic Kähler (briefly QK). This condition is equivalent to the statement that the holonomy group of g is contained in $\mathrm{SP}(n) \cdot \mathrm{SP}(1)$ [1,2,25,40,41]. If on a QK manifold there exist an admissible basis (H) such that each almost complex structure $(J_\alpha) \in (H), \alpha = 1, 2, 3$ is parallel with respect to the Levi-Civita connection then the manifold is called hyper Kähler (briefly HK). In this case, the holonomy group of g is contained in $\mathrm{SP}(n)$.

The notions of quaternionic manifolds arise in a natural way from the theory of supersymmetric sigma models. The geometry of the target space of two-dimensional sigma models with extended supersymmetry is described by the properties of a metric connection with torsion [14,22]. The geometry of $(4, 0)$ supersymmetric two-dimensional sigma models without Wess–Zumino term (torsion) is a hyper Kähler manifold. In the presence of torsion the geometry of the target space becomes hyper Kähler with torsion (briefly HKT) [23]. This means that the complex structures $J_\alpha, \alpha = 1, 2, 3$, are parallel with respect to a metric quaternionic connection with totally skew-symmetric torsion [23]. Local $(4, 0)$ supersymmetry requires that the target space of two-dimensional sigma models with Wess–Zumino term be either HKT or quaternionic Kähler with torsion (briefly QKT) [32] which means that the quaternionic subbundle is parallel with respect to a metric linear connection with totally skew-symmetric torsion and the torsion 3-form is of type $(1, 2) + (2, 1)$ with respect to all almost complex structures in Q. The target space of two-dimensional $(4, 0)$ supersymmetric sigma models with torsion coupled to $(4, 0)$ supergravity is a QKT manifold [24]. If the torsion of a QKT manifold is a closed 3-form then it is called strong QKT manifold. The properties of HKT and QKT geometries strongly resemble those of HK and QK ones, respectively. In particular, HKT [23] and QKT [24] manifolds admit twistor constructions with twistor spaces which have similar properties to those of HK [21] and QK [40–42].

The main object of interest in this paper is the differential geometric properties of QKT manifolds. We find necessary and sufficient conditions to the existence of a QKT connection in terms of the Kähler 2-forms and show that the QKT connection is unique if dimension is at least 8 (see Theorem 2.2). We prove that the QKT manifolds are invariant under conformal transformations of the metric. This allows us to present a lot of (compact) examples of QKT manifolds. In particular, we show that the compact quaternionic Hopf manifolds studied in

[35], which do not admit a QK structure, are QKT manifolds. In the compact case, we show the existence of Gauduchon metric, i.e. the unique conformally equivalent QKT structure with co-closed torsion 1-form.

It is shown in [24] that the twistor space of a QKT manifold is always complex manifold provided the dimension is at least 8. It admits complex contact (resp. Kähler) structure if the torsion 4-form is of type $(2, 2)$ and some additional non-degenerativity (positivity) conditions are fulfilled [24]. Most of the known examples of QKT manifolds are homogeneous constructed in [34]. However, there are no homogeneous proper QKT manifolds (i.e. QKT which is not QK or HKT) with torsion 4-form of type $(2, 2)$ in dimensions greater than 4 by the result of Opfermann and Papadopoulos [34]. We generalise this result showing that there are no proper QKT manifolds with torsion 4-form of type $(2, 2)$ provided that the torsion is parallel and dimension is at least 8.

In dimension 4, a lot of examples of QKT manifolds are known [24,34]. In particular, examples of homogeneous QKT manifolds are constructed in [34]. We notice that there are many (even strong) QKT structures in dimension 4, all depending on an arbitrary 1-form. We give a local description of four-dimensional QKT manifolds with parallel torsion; namely such a QKT manifold is a Riemannian product of a real line and a three-dimensional Riemannian manifold. We observe that homogeneous QKT manifolds are precisely naturally reductive homogeneous Riemannian manifolds, the objects which are well known. We present a complete local description (up to an isometry) of four-dimensional homogeneous QKT which was known in the setting of naturally reductive homogeneous 4-manifold [27]. In the last section, we consider four-dimensional Einstein-like QKT manifold and find a closed relation with Einstein–Weyl geometry in dimension 4. In particular, we show that every four-dimensional HKT manifold is of this type.

2. Characterisations of QKT connection

Let $(M, g, (J_\alpha) \in Q, \alpha = 1, 2, 3)$ be a $4n$-dimensional almost quaternionic manifold with Q-Hermitian Riemannian metric g and an admissible basis (J_α). The Kähler form F_α of each J_α is defined by $F_\alpha = g(\cdot, J_\alpha)$. The corresponding Lie forms are given by $\theta_\alpha = \delta F_\alpha \circ J_\alpha$.

For an r-form ψ, we denote by $J_\alpha\psi$ the r-form defined by $J_\alpha\psi(X_1, \dots, X_r) := (-1)^r\psi(J_\alpha X_1, \dots, J_\alpha X_r), \alpha = 1, 2, 3$. Then $(\mathrm{d}^c\psi)_\alpha = (-1)^r J_\alpha\, \mathrm{d}J_\alpha\psi$. We shall use the notations $\mathrm{d}_\alpha F_\beta := (\mathrm{d}^c F_\beta)_\alpha$, i.e. $\mathrm{d}_\alpha F_\beta(X, Y, Z) = -\mathrm{d}F_\beta(J_\alpha X, J_\alpha Y, J_\alpha Z), \alpha, \beta = 1, 2, 3$.

We recall the decomposition of a skew-symmetric tensor $P \in \Lambda^2 T^*M \otimes TM$ with respect to a given almost complex structure J_α. The $(1, 1)$-, $(2, 0)$- and $(0, 2)$-part of P are defined by $P^{1,1}(J_\alpha X, J_\alpha Y) = P^{1,1}(X, Y)$, $P^{2,0}(J_\alpha X, Y) = J_\alpha P^{2,0}(X, Y)$, $P^{0,2}(J_\alpha X, Y) = -J_\alpha P^{0,2}(X, Y)$, respectively.

For each $\alpha = 1, 2, 3$, we denote by $\mathrm{d}F_\alpha^+$ (resp. $\mathrm{d}F_\alpha^-$), the $(1, 2)+(2, 1)$-part (resp. $(3, 0)+(0, 3)$-part) of $\mathrm{d}F_\alpha$ with respect to the almost complex structure J_α. We consider the following 1-form:

$$\theta_{\alpha,\beta} = -\frac{1}{2}\sum_{i=1}^{4n}\mathrm{d}F_\alpha^+(X, e_i, J_\beta e_i), \quad \alpha, \beta = 1, 2, 3.$$

Here and further $e_1, e_2, \dots, 4n$ is an orthonormal basis of the tangential space.

Note that $\theta_{\alpha,\alpha} = \theta_\alpha$.

The Nijenhuis tensor N_α of an almost complex structure J_α is given by $N_\alpha(X, Y) = [J_\alpha X, J_\alpha Y] - [X, Y] - J_\alpha[J_\alpha X, Y] - J_\alpha[X, J_\alpha Y]$.

The celebrated Newlander–Nirenberg theorem [31] states that an almost complex structure is a complex structure if and only if its Nijenhuis tensor vanishes.

Let ∇ be a quaternionic connection, i.e.

$$\nabla J_\alpha = -\omega_\beta \otimes J_\gamma + \omega_\gamma \otimes J_\beta, \tag{2.1}$$

where the $\omega_\alpha, \alpha = 1, 2, 3$ are 1-forms.

Here and henceforth (α, β, γ) is a cyclic permutation of $(1, 2, 3)$.

Let $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ be the torsion tensor of type $(1, 2)$ of ∇. We denote by the same letter the torsion tensor of type $(0, 3)$ given by $T(X, Y, Z) = g(T(X, Y), Z)$. The Nijenhuis tensor is expressed in terms of ∇ as follows:

$$\begin{aligned} N_\alpha(X, Y) = {} & 4T_\alpha^{0,2}(X, Y) + (\nabla_{J_\alpha X} J_\alpha)(Y) - (\nabla_{J_\alpha Y} J_\alpha)(X) \\ & - (\nabla_Y J_\alpha)(J_\alpha X) + (\nabla_X J_\alpha)(J_\alpha Y), \end{aligned} \tag{2.2}$$

where the $(0, 2)$-part $T_\alpha^{0,2}$ of the torsion with respect to J_α is given by

$$T_\alpha^{0,2}(X, Y) = \tfrac{1}{4}(T(X, Y) - T(J_\alpha X, J_\alpha Y) + J_\alpha T(J_\alpha X, Y) + J_\alpha T(X, J_\alpha Y)). \tag{2.3}$$

We recall that if a 3-form ψ is of type $(1, 2) + (2, 1)$ with respect to an almost complex structure J then it satisfies the equality

$$\psi(X, Y, Z) = \psi(JX, JY, Z) + \psi(JX, Y, JZ) + \psi(X, JY, JZ). \tag{2.4}$$

Definition. An almost quaternionic Hermitian manifold $(M, g, (H_\alpha) \in Q)$ is *QKT manifold* if it admits a metric quaternionic connection ∇ with totally skew-symmetric torsion which is $(1, 2) + (2, 1)$-form with respect to each $J_\alpha, \alpha = 1, 2, 3$. If the torsion 3-form is closed then the manifold is said to be *strong QKT manifold*.

It follows that the holonomy group of ∇ is a subgroup of $\mathrm{SP}(n) \cdot \mathrm{SP}(1)$.

By means of (2.1), (2.2) and (2.4), the Nijenhuis tensor N_α of $J_\alpha, \alpha = 1, 2, 3$, on a QKT manifold is given by

$$N_\alpha(X, Y) = A_\alpha(Y) J_\beta X - A_\alpha(X) J_\beta Y - J_\alpha A_\alpha(Y) J_\gamma X + J_\alpha A_\alpha(X) J_\gamma Y, \tag{2.5}$$

where

$$A_\alpha = \omega_\beta + J_\alpha \omega_\gamma. \tag{2.6}$$

Remark 2.1. The definition of QKT manifolds given above is equivalent to that given in [24] because the requirement the torsion to be $(1, 2) + (2, 1)$-form with respect to each $J_\alpha, \alpha = 1, 2, 3$, is equivalent, by means of (2.5), to the fourth condition of (4) in [24]. The torsion of ∇ is $(1, 2) + (2, 1)$-form with respect to any (local) almost complex structure

$J \in Q$ [24]. This follows also from (2.5) and the general formula (6) in [4] which expresses N_J in terms of $N_{J_1}, N_{J_2}, N_{J_3}$. In fact, it is sufficient that the torsion is a $(1,2)+(2,1)$-form with respect to the only two almost complex structures of (H) since the formula (3.4.4) in [3] gives the necessary expression of N_{J_3} by N_{J_1} and N_{J_2}. Indeed, it is easy to see that the formula (3.4.4) in [3] holds for the $(0,2)$-part $T^{0,2}_\alpha$, $\alpha = 1,2,3$, of the torsion. Hence, the vanishing of the $(0,2)$-part of the torsion with respect to any two almost complex structures in (H) implies the vanishing of the $(0,2)$-part of T with respect to the third one.

On a QKT manifold there are three naturally associated 1-forms to the torsion defined by

$$t_\alpha(X) = -\frac{1}{2}\sum_{i=1}^{4n} T(X, e_i, J_\alpha e_i), \quad \alpha = 1,2,3. \tag{2.7}$$

We have the following proposition.

Proposition 2.1. *On a QKT manifold,* $J_1 t_1 = J_2 t_2 = J_3 t_3$.

Proof. Applying (2.4) with respect to J_β, we obtain

$$\begin{aligned} t_\alpha(X) &= -\frac{1}{2}\sum_{i=1}^{4n} T(X, e_i, J_\alpha e_i) = -\frac{1}{2}\sum_{i=1}^{4n} T(X, J_\beta e_i, J_\gamma e_i) \\ &= \frac{1}{2}\sum_{i=1}^{4n} T(J_\beta X, e_i, J_\gamma e_i) - \frac{1}{2}\sum_{i=1}^{4n} T(J_\beta X, J_\beta e_i, J_\alpha e_i) + \frac{1}{2}\sum_{i=1}^{4n} T(X, e_i, J_\alpha e_i). \end{aligned}$$

The last equality implies $t_\alpha = J_\beta t_\gamma$ which proves the assertion. □

The 1-form $t = J_\alpha t_\alpha$ is independent of the chosen almost complex structure J_α by Proposition 2.1. We shall call it *the torsion 1-form* of a given QKT manifold.

Remark 2.2. Every QKT manifold is a quaternionic manifold. This is an immediate consequence of (2.5) and Proposition 2.3 in [4].

However, the converse to the above property is not always true. In fact, we have the following theorem.

Theorem 2.2. *Let* $(M, g, (J_\alpha) \in Q)$ *be a* $4n$*-dimensional* $(n > 1)$ *quaternionic manifold with* Q*-Hermitian metric* g. *Then* M *admits a QKT structure if and only if the following conditions hold*:

$$(d_\alpha F_\alpha)^+ - (d_\beta F_\beta)^+ = \tfrac{1}{2}(K_\alpha \wedge F_\beta - J_\beta K_\beta \wedge F_\alpha - (K_\beta - J_\alpha K_\alpha) \wedge F_\gamma), \tag{2.8}$$

where $(d_\alpha F_\alpha)^+$ *denotes the* $(1,2)+(2,1)$*-part of* $(d_\alpha F_\alpha)$ *with respect to the* J_α, $\alpha = 1,2,3$. *The 1-forms* K_α, $\alpha = 1,2,3$, *are given by*

$$K_\alpha = \frac{1}{1-n}(J_\beta \theta_\alpha + \theta_{\alpha,\gamma}). \tag{2.9}$$

The metric quaternionic connection ∇ *with torsion* 3*-form of type* $(1,2)+(2,1)$ *is unique and is determined by*

$$\nabla = \nabla^g + \tfrac{1}{2}((\mathrm{d}_\alpha F_\alpha)^+ - \tfrac{1}{2}(J_\alpha K_\alpha \wedge F_\gamma + K_\alpha \wedge F_\beta)), \tag{2.10}$$

where ∇^g *is the Levi-Civita connection of* g.

Proof. To prove the 'if' part, let ∇ be a metric quaternionic connection satisfying (2.1) which torsion T has the required properties. We follow the scheme in [17]. Since T is skew-symmetric, we have

$$\nabla = \nabla^g + \tfrac{1}{2}T. \tag{2.11}$$

We obtain using (2.1) and (2.11) that

$$\begin{aligned}&\tfrac{1}{2}(T(X, J_\alpha Y, Z) + T(X, Y, J_\alpha Z))\\ &\quad = -g((\nabla^g_X J_\alpha)Y, Z) + \omega_\beta(X)F_\gamma(Y,Z) - \omega_\gamma(X)F_\beta(Y,Z).\end{aligned} \tag{2.12}$$

The tensor $\nabla^g J_\alpha$ is decomposed by parts according to $\nabla J_\alpha = (\nabla J_\alpha)^{2,0} + (\nabla J_\alpha)^{0,2}$, where [17]

$$g((\nabla^g_X J_\alpha)^{2,0}Y, Z) = \tfrac{1}{2}((\mathrm{d}F_\alpha)^+(X, J_\alpha Y, J_\alpha Z) - (\mathrm{d}F_\alpha)^+(X,Y,Z)), \tag{2.13}$$

$$\begin{aligned}g((\nabla^g_X J_\alpha)^{0,2}Y, Z) = \tfrac{1}{2}(&g(N_\alpha(X,Y), J_\alpha Z) - g(N_\alpha(X,Z), J_\alpha Y)\\ &- g(N_\alpha(Y,Z), J_\alpha X)).\end{aligned} \tag{2.14}$$

Taking the $(2,0)$-part in (2.12), we obtain using (2.13) that

$$\begin{aligned}T(X, J_\alpha Y, Z) + T(X, Y, J_\alpha Y) = {}&(\mathrm{d}F_\alpha)^+(X, J_\alpha Y, J_\alpha Z) - (\mathrm{d}F_\alpha)^+(X,Y,Z)\\ &+ C_\alpha(X)F_\gamma(Y,Z) + C_\alpha(J_\alpha X)F_\beta(Y,Z),\end{aligned} \tag{2.15}$$

where

$$C_\alpha = \omega_\beta - J_\alpha \omega_\gamma. \tag{2.16}$$

The cyclic sum of (2.15) and the fact that T and $(\mathrm{d}F_\alpha)^+$ are $(1,2)+(2,1)$-forms with respect to each J_α, gives

$$T = (\mathrm{d}_\alpha F_\alpha)^+ - \tfrac{1}{2}(J_\alpha C_\alpha \wedge F_\gamma + C_\alpha \wedge F_\beta). \tag{2.17}$$

Further, we take the contractions in (2.17) to get

$$\begin{aligned}&J_\alpha t_\alpha = -\theta_\alpha - J_\beta C_\alpha, \qquad J_\alpha t_\alpha = -J_\gamma \theta_{\beta,\alpha} - nJ_\gamma C_\beta,\\ &J_\alpha t_\alpha = J_\beta \theta_{\gamma,\alpha} - nJ_\alpha C_\gamma.\end{aligned} \tag{2.18}$$

Using Proposition 2.1, (2.6) and (2.16), we obtain consequently from (2.18) that

$$A_\alpha = J_\alpha C_\beta + J_\gamma C_\gamma = J_\beta(\theta_\gamma - \theta_\beta), \tag{2.19}$$

$$(n-1)J_\beta C_\alpha = \theta_\alpha - J_\beta \theta_{\alpha,\gamma}. \tag{2.20}$$

Then (2.8) and (2.9) follow from (2.17) and (2.20).

For the converse, we define ∇ by (2.10). To complete the proof, we have to show that ∇ is a quaternionic connection. We calculate

$$g((\nabla_X J_\alpha)Y, Z) = g((\nabla^g_X J_\alpha)Y, Z) + \tfrac{1}{2}(T(X, J_\alpha Y, Z) + T(X, Y, J_\alpha Z))$$
$$= \omega_\beta(X)F_\gamma(Y, Z) - \omega_\gamma(X)F_\beta(Y, Z),$$

where we used (2.6), (2.9), (2.13), (2.14), (2.16) and (2.19) and the compatibility condition (2.8) to get the last equality. The uniqueness of ∇ follows from (2.10) as well as from Theorem 10.3 in [33] which states that any quaternionic connection is entirely determined by its torsion (see also [18]). □

In the case of HKT manifold, $K_\alpha = \mathrm{d}F^-_\alpha = 0$ and Theorem 2.2 is a consequence of the general results in [17] (see also [20]) which imply that on a Hermitian manifold there exists a unique linear connection with totally skew-symmetric torsion preserving the metric and the complex structure, the Bismut connection. This connection was used by Bismut [9] to prove a local index theorem for the Dolbeault operator on non-Kähler manifold. The geometry of this connection is referred to KT-geometry by physicists. Obstructions to the existence of (non-trivial) Dolbeault cohomology groups on a compact KT-manifold are presented in [5].

We note that (2.19) and (2.20) are also valid in the case $n = 1$.

We get, as a consequence of the proof of Theorem 2.2, the following integrability criterion which is discovered in dimension 4 by Battaglia and Salamon (see [19]).

Proposition 2.3. *The Nijenhuis tensors of a QKT manifold depend only on the difference between the Lie forms. In particular, the almost complex structures J_α on a QKT manifold $(M, (J_\alpha) \in Q, g, \nabla)$ are integrable if and only if*

$$\theta_\alpha = \theta_\beta = \theta_\gamma.$$

Proof. The Nijenhuis tensors are given by (2.5) and (2.19). □

Corollary 2.4. *On a 4n-dimensional QKT manifold the following formulas hold*:

$$J_\beta\theta_{\alpha,\gamma} = -J_\gamma\theta_{\alpha,\beta},$$
$$(n^2 + n)\theta_\alpha - n\theta_\beta - n^2\theta_\gamma + J_\gamma\theta_{\beta,\alpha} + nJ_\alpha\theta_{\gamma,\beta} - (n+1)J_\beta\theta_{\alpha,\gamma} = 0. \tag{2.21}$$

If $n = 1$, then

$$\theta_\alpha = J_\beta\theta_{\alpha,\gamma} = -J_\gamma\theta_{\alpha,\beta}.$$

Proof. The first formula follows directly from the system (2.18). Solving the system (2.18) with respect to C_α, we obtain

$$(n^3 - 1)J_\beta C_\alpha = (\theta_\alpha - J_\gamma\theta_{\beta,\alpha}) + n(\theta_\beta - J_\alpha\theta_{\gamma,\beta}) + n^2(\theta_\gamma - J_\beta\theta_{\alpha,\gamma}). \tag{2.22}$$

Then (2.21) is a consequence of (2.20) and (2.22). The last assertion follows from (2.20). □

Corollary 2.5. *On a* $4n$*-dimensional* $(n > 1)$ *QKT manifold the* $\mathrm{SP}(1)$*-connection* 1*-forms are given by*

$$\omega_\beta = \frac{1}{2} J_\beta \left(\theta_\gamma - \theta_\beta + \frac{1}{1-n}\theta_\alpha\right) + \frac{1}{2(1-n)}\theta_{\alpha,\gamma}. \tag{2.23}$$

Proof. The proof follows in a straightforward way from (2.6), (2.16), (2.19) and (2.20). □

Theorem 2.2 and the above formulas lead to the following criterion.

Proposition 2.6. *Let* $(M, g, (H))$ *be a* $4n$*-dimensional* $(n > 1)$ *QKT manifold. The following conditions are equivalent:*

1. $(M, g, (H))$ *is an HKT manifold*;
2. $\mathrm{d}_\alpha F^+_\alpha = \mathrm{d}_\beta F^+_\beta = \mathrm{d}_\gamma F^+_\gamma$;
3. $\theta_\alpha = J_\beta \theta_{\gamma,\alpha}$.

Proof. If $(M, g, (H))$ is an HKT manifold, the connection 1-forms $\omega_\alpha = 0, \alpha = 1, 2, 3$. Then (2) and (3) follow from (2.8), (2.9), (2.16) and (2.20).

If (3) holds, then (2.19) and (2.20) yield $C_\alpha = A_\alpha = 0, \alpha = 1, 2, 3$, since $n > 1$. Consequently, $2\omega_\alpha = J_\beta C_\beta - J_\beta A_\beta = 0$ by (2.6) and (2.16). Thus the equivalence of (1) and (3) is proved.

Let (2) holds, then we compute that $\theta_\alpha = J_\gamma \theta_{\beta,\alpha}$. Since $n > 1$, the equality (2.22) leads to $C_\alpha = 0, \alpha = 1, 2, 3$, which forces $\omega_\alpha = 0, \alpha = 1, 2, 3$ as above. This completes the proof. □

The next theorem shows that QKT manifolds are stable under a conformal transformations.

Theorem 2.7. *Let* $(M, g, (J_\alpha), \nabla)$ *be a* $4n$*-dimensional QKT manifold. Then every Riemannian metric* $\bar{g}$ *in the conformal class* $[g]$ *admits a QKT connection. If* $\bar{g} = fg$ *for a positive function* f *then the QKT connection* $\bar{\nabla}$ *corresponding to* $\bar{g}$ *is given by*

$$\begin{aligned}\bar{g}(\bar{\nabla}_X Y, Z) = {}& fg(\nabla_X Y, Z) + \tfrac{1}{2}(\mathrm{d}f(X)g(Y, Z) + \mathrm{d}f(Y)g(X, Z) - \mathrm{d}f(Z)g(X, Y)) \\ & + \tfrac{1}{2}(J_\alpha\, \mathrm{d}f \wedge F_\alpha + J_\beta\, \mathrm{d}f \wedge F_\beta + J_\gamma\, \mathrm{d}f \wedge F_\gamma)(X, Y, Z). \end{aligned} \tag{2.24}$$

The torsion tensors T *and* $\bar{T}$ *and the torsion* 1*-forms* t *and* $\bar{t}$ *of* ∇ *and* $\bar{\nabla}$ *are related by*

$$\bar{T} = fT + J_\alpha\, \mathrm{d}f \wedge F_\alpha + J_\beta\, \mathrm{d}f \wedge F_\beta + J_\gamma\, \mathrm{d}f \wedge F_\gamma, \tag{2.25}$$

$$\bar{t} = t - (2n+1)\, \mathrm{d} \ln f. \tag{2.26}$$

Proof. First, we assume $n > 1$. We shall apply Theorem 2.2 to the quaternionic Hermitian manifold $(M, \bar{g} = fg, (J_\alpha) \in Q)$. We denote the objects corresponding to the metric $\bar{g}$ by a line above the symbol, e.g. $\bar{F}_\alpha$ denotes the Kähler form of J_α with respect to $\bar{g}$. An easy calculation gives the following sequence of formulas:

$$\mathrm{d}_\alpha \bar{F}^+_\alpha = J_\alpha\, \mathrm{d}f \wedge F_\alpha + f\, \mathrm{d}_\alpha F^+_\alpha, \qquad \bar{\theta}_\alpha = \theta_\alpha + (2n-1)\, \mathrm{d}\ln f,$$
$$\bar{\theta}_{\alpha,\gamma} = \theta_{\alpha,\gamma} - J_\beta\, \mathrm{d}\ln f. \tag{2.27}$$

We substitute (2.27) into (2.9), (2.19) and (2.23) to get

$$\bar{K}_\alpha = K_\alpha - 2J_\beta\, \mathrm{d}\ln f, \qquad \bar{A} = A, \qquad \bar{\omega}_\alpha = \omega_\alpha - J_\beta\, \mathrm{d}\ln f. \tag{2.28}$$

Using (2.27) and (2.28), we verify that the conditions (2.8) with respect to the metric $\bar{g}$ are fulfilled. Theorem 2.2 implies that there exists a QKT connection $\bar{\nabla}$ with respect to $(\bar{g}, Q)$. Using the well-known relation between the Levi-Civita connections of conformally equivalent metrics, (2.27) and (2.28), we obtain (2.24) from (2.10).

If $n = 1$, we define the new QKT connection with respect to $(\bar{g}, Q)$ by (2.24).

Using (2.24), we get (2.25) and consequently (2.26). □

Theorem 2.7 allows us to find distinguished QKT structures on a compact QKT manifold. To this end, we shall use the Gauduchon theorem for the existence of a Gauduchon metric on a compact Hermitian or Weyl manifold [15,16]. This theorem can be formulated in our notations as follows: to a given compact QKT manifold $(M, g, (J_\alpha), \nabla, T)$ there always exists a unique (up to homothety) conformally related QKT manifold $(M, g_G = fg, (J_\alpha), \nabla_G, T_G)$ such that the corresponding torsion 1-form t_G is co-closed with respect to g_G. The key point is that the torsion 1-form transforms under conformal rescaling according to (2.26) (see [44, Appendix 1]). Application of this theorem leads to the following theorem.

Theorem 2.8. *In the conformal class of a compact QKT manifold there exists a unique* (*up to homothety*) *metric with co-closed torsion* 1*-form*.

We shall call the metric with co-closed torsion 1-form on a compact QKT manifold the *Gauduchon metric*.

Corollary 2.9. *On a compact QKT manifold with closed* (*non-exact*) *torsion* 1*-form the Gauduchon metric* g_G *cannot have positive definite Riemannian Ricci tensor. In particular, if it is a Einstein manifold then it is of non-positive scalar curvature.*

Further, if the Gauduchon metric is Ricci flat then the corresponding torsion 1*-form* t_G *is parallel with respect to the Levi-Civita connection of* g_G.

Proof. The two form $\mathrm{d}t$ is invariant under conformal transformations by (2.26). Then the Gauduchon metric has harmonic torsion 1-form, i.e. $\mathrm{d}t = \delta t = 0$. The claim follows from the Weitzenböeck formula (see, e.g. [8]) $\int_M \{|\mathrm{d}t|^2 + |\delta t|^2\}\, \mathrm{d}V = \int_M \{|\nabla^g t|^2 + \mathrm{Ric}^g(t^\#, t^\#)\}\, \mathrm{d}V = 0$, where $t^\#$ is the dual vector field of t, $|\cdot|$ is the usual tensor norm and $\mathrm{d}V$ the volume form. □

Theorem 2.7 allows us to supply a large class of (compact) QKT manifold. Namely, any conformal metric of a QK, HK or HKT manifold will give a QKT manifold. This leads to the notion of *locally conformally QK* (*resp. locally conformally HK, resp. locally conformally HKT*) *manifolds* (briefly l.c.QK (resp. l.c.HK, resp. l.c.HKT) manifolds) in the context of QKT geometry.

The l.c.QK and l.c.HK manifolds have already appeared in the context of Hermitian–Einstein–Weyl structures [37] and of 3-Sasakian structures [12]. These two classes of quaternionic manifolds are studied in detail (mostly in the compact case) in [35,36].

We recall that a quaternionic Hermitian manifold (M, g, Q) is said to be l.c.QK (resp. l.c. HK, resp. l.c.HKT) manifold if each point $p \in M$ has a neighbourhood U_p such that $g|_{U_p}$ is conformally equivalent to a QK (resp. HK, resp. HKT) metric. There are compact l.c.QK manifold which do not admit any QK structure [35]. Typical examples of compact l.c.QK manifolds without any QK structure are the quaternionic Hopf spaces $H = (\mathcal{H}^n - \{0\})/\Gamma$, where Γ is an appropriate discrete group acting diagonally on the quaternionic coordinates in $\mathcal{H}^n$ (see [35]).

We recall that on a l.c.QK manifold the 4-form $\Omega = \sum_{\alpha=1}^{3} F_\alpha \wedge F_\alpha$ satisfies $\mathrm{d}\Omega = \omega \wedge \Omega$, $\mathrm{d}\omega = 0$, where ω is locally defined by $\omega = 2\,\mathrm{d}\ln f$. On a l.c.QK manifold viewed as a QKT manifold by Theorem 2.7 the torsion 1-form is equal to $t = (2n+1)\omega$ by (2.26). The QK manifolds are Einstein provided the dimension is at least 8 [1,7]. Then, the Gauduchon theorem [16] applied to l.c.QK manifold in [35] can be stated in our context as follows.

Corollary 2.10. *Let (M, g) be a compact $4n$-dimensional $(n > 1)$ QKT manifold which is l.c.QK and assume that no metric in the conformal class $[g]$ of g is QK. Then the torsion 1-form of the Gauduchon metric g_G is parallel with respect to the Levi-Civita connection of g_G.*

Theorems 2.2 and 2.7 together with Propositions 2.3 and 2.6 imply the following.

Corollary 2.11. *Every l.c.QK manifold admits a QKT structure.*
Further, if $(M, g, (J_\alpha), \nabla)$ is a $4n$-dimensional $(n > 1)$ QKT manifold then:

1. $(M, g, (J_\alpha), \nabla)$ *is a l.c.QK manifold if and only if*
$$T = \frac{1}{2n+1}(t_\alpha \wedge F_\alpha + t_\beta \wedge F_\beta + t_\gamma \wedge F_\gamma), \quad \mathrm{d}t = 0; \tag{2.29}$$
2. $(M, g, (J_\alpha), \nabla)$ *is a l.c.HKT manifold if and only if the 1-form $\theta_\alpha - J_\beta\theta_{\alpha,\gamma}$ is closed, i.e.*
$$\mathrm{d}(\theta_\alpha - J_\beta\theta_{\alpha,\gamma}) = 0;$$
3. $(M, g, (J_\alpha), \nabla)$ *is a l.c.HK manifold if an only if* (2.29) *holds and*
$$\theta_\alpha - J_\beta\theta_{\alpha,\gamma} = \frac{2(1-n)}{2n+1}t.$$

3. Curvature of a QKT space

Let $R = [\nabla, \nabla] - \nabla_{[,]}$ be the curvature tensor of type $(1, 3)$ of ∇. We denote the curvature tensor of type $(0, 4)$ $R(X, Y, Z, V) = g(R(X, Y)Z, V)$ by the same letter. There are three

Ricci forms given by

$$\rho_\alpha(X,Y)=\frac{1}{2}\sum_{i=1}^{4n}R(X,Y,e_i,J_\alpha e_i),\quad \alpha=1,2,3.$$

Proposition 3.1. *The curvature of a QKT manifold* $(M,g,(J_\alpha),\nabla)$ *satisfies the following relations*:

$$R(X,Y)J_\alpha=\frac{1}{n}(\rho_\gamma(X,Y)J_\beta-\rho_\beta(X,Y)J_\gamma), \tag{3.30}$$

$$\rho_\alpha=\mathrm{d}\omega_\alpha+\omega_\beta\wedge\omega_\gamma. \tag{3.31}$$

Proof. We follow the classical scheme (see, e.g. [3,8,36]). Using (2.1), we obtain

$$R(X,Y)J_\alpha=-(\mathrm{d}\omega_\beta+\omega_\gamma\wedge\omega_\alpha)(X,Y)J_\gamma+(\mathrm{d}\omega_\gamma+\omega_\alpha\wedge\omega_\beta)(X,Y)J_\beta.$$

Taking the trace in the last equality, we get

$$\rho_\alpha(X,Y)=\frac{1}{2}\sum_{i=1}^{4n}R(X,Y,e_i,J_\alpha e_i)=\frac{1}{2}\sum_{i=1}^{4n}R(X,Y,J_\beta e_i,J_\gamma e_i)$$
$$=-\frac{1}{2}\sum_{i=1}^{4n}R(X,Y,e_i,J_\alpha e_i)+2n(\mathrm{d}\omega_\alpha+\omega_\beta\wedge\omega_\gamma)(X,Y)J_\beta. \qquad \square$$

Using Proposition 3.1, we find a simple necessary and sufficient condition a QKT manifold to be an HKT one, i.e. the holonomy group of ∇ to be a subgroup of $\mathrm{SP}(n)$.

Proposition 3.2. *A* $4n$*-dimensional* $(n>1)$ *QKT manifold is an HKT manifold if and only if all the three Ricci forms vanish, i.e.* $\rho_1=\rho_2=\rho_3=0$.

Proof. If a QKT manifold is an HKT manifold then the holonomy group of ∇ is contained in $\mathrm{SP}(n)$. This implies $\rho_\alpha=0$, $\alpha=1,2,3$.

For the converse, let the three Ricci forms vanish. Eq. (3.31) mean that the curvature of the SP(1) connection on Q vanish. Then there exists a basis $(I_\alpha,\alpha=1,2,3)$ of almost complex structures on Q and each I_α is ∇-parallel, i.e. the corresponding connection 1-forms $\omega_{I_\alpha}=0$, $\alpha=1,2,3$. Then each I_α is a complex structure, by (2.5) and (2.6). This implies that the QKT manifold is an HKT manifold. $\square$

We denote by Ric, Ric^g the Ricci tensors of the QKT connection and of the Levi-Civita connection, respectively. In fact $\mathrm{Ric}(X,Y)=\sum_{i=1}^{4n}R(e_i,X,Y,e_i)$.

Our main technical result is the following proposition.

Proposition 3.3. *Let* $(M,g,(J_\alpha),\nabla)$ *be a* $4n$*-dimensional QKT manifold. The following formulas hold*:

$$
\begin{aligned}
&n\rho_\alpha(X, J_\alpha Y) + \rho_\beta(X, J_\beta Y) + \rho_\gamma(X, J_\gamma Y) \\
&\quad = -n \operatorname{Ric}(XY) + \tfrac{1}{4}n(\mathrm{d}T)_\alpha(X, J_\alpha Y) + \tfrac{1}{2}n(\nabla T)_\alpha(X, J_\alpha Y),
\end{aligned} \tag{3.32}
$$

$$
\begin{aligned}
(n-1)\rho_\alpha(X, J_\alpha Y) = &-\frac{n(n-1)}{n+2}\operatorname{Ric}(X, Y) + \frac{n}{4(n+2)}\{(n+1)(\mathrm{d}T)_\alpha(X, J_\alpha Y) \\
&- (\mathrm{d}T)_\beta(X, J_\beta Y) - (\mathrm{d}T)_\gamma(X, J_\gamma Y)\}, \\
&+ \frac{n}{2(n+2)}\{(n+1)(\nabla T)_\alpha(X, J_\alpha Y) - (\nabla T)_\beta(X, J_\beta Y) \\
&- (\nabla T)_\gamma(X, J_\gamma Y)\},
\end{aligned} \tag{3.33}
$$

where $(\mathrm{d}T)_\alpha(X, Y) = \sum_{i=1}^{4n}\mathrm{d}T(X, Y, e_i, J_\alpha e_i)$, $(\nabla T)_\alpha(X, Y) = \sum_{i=1}^{4n}(\nabla_X T)(Y, e_i, J_\alpha e_i)$.

Proof. Since the torsion is a 3-form, we have

$$
(\nabla^g_X T)(Y, Z, U) = (\nabla_X T)(Y, Z, U) + \frac{1}{2}\underset{XYZ}{\sigma}\{g(T(X, Y), T(Z, U))\}, \tag{3.34}
$$

where $\underset{XYZ}{\sigma}$ denote the cyclic sum of X, Y, Z.

The exterior derivative $\mathrm{d}T$ is given by

$$
\begin{aligned}
\mathrm{d}T(X, Y, Z, U) = &\underset{XYZ}{\sigma}\{(\nabla_X T)(Y, Z, U) + g(T(X, Y), T(Z, U))\} \\
&- (\nabla_U T)(X, Y, Z) + \underset{XYZ}{\sigma}\{g(T(X, Y), T(Z, U))\}.
\end{aligned} \tag{3.35}
$$

The first Bianchi identity for ∇ states

$$
\underset{XYZ}{\sigma} R(X, Y, Z, U) = \underset{XYZ}{\sigma}\{(\nabla_X T)(Y, Z, U) + g(T(X, Y), T(Z, U))\}. \tag{3.36}
$$

We denote by B the Bianchi projector, i.e. $B(X, Y, Z, U) = \underset{XYZ}{\sigma} R(X, Y, Z, U)$.

The curvature R^g of the Levi-Civita connection is connected by R in the following way:

$$
\begin{aligned}
R^g(X, Y, Z, U) = &R(X, Y, Z, U) - \tfrac{1}{2}(\nabla_X T)(Y, Z, U) + \tfrac{1}{2}(\nabla_Y T)(X, Z, U) \\
&- \tfrac{1}{2}g(T(X, Y), T(Z, U)) - \tfrac{1}{4}g(T(Y, Z), T(X, U)) \\
&- \tfrac{1}{4}g(T(Z, X), T(Y, U)).
\end{aligned} \tag{3.37}
$$

Define D by $D(X, Y, Z, U) = R(X, Y, Z, U) - R(Z, U, X, Y)$, we obtain from (3.37)

$$
\begin{aligned}
D(X, Y, Z, U) = &\tfrac{1}{2}(\nabla_X T)(Y, Z, U) - \tfrac{1}{2}(\nabla_Y T)(X, Z, U) \\
&- \tfrac{1}{2}(\nabla_Z T)(U, X, Y) + \tfrac{1}{2}(\nabla_U T)(Z, X, Y),
\end{aligned} \tag{3.38}
$$

since D^g of R^g is zero.

Using (3.30) and (3.36), we find the following relation between the Ricci tensor and the Ricci forms:

$$\begin{aligned}\rho_\alpha(X,Y) &= -\frac{1}{2}\sum_{i=1}^{4n}(R(Y,e_i,X,J_\alpha e_i)+R(e_i,X,Y,J_\alpha e_i))+\frac{1}{2}\sum_{i=1}^{4n}B(X,Y,e_i,J_\alpha e_i)\\ &= -\frac{1}{2}\,\mathrm{Ric}(Y,J_\alpha X)+\frac{1}{2}\,\mathrm{Ric}(X,J_\alpha Y)+\frac{1}{2}\sum_{i=1}^{4n}B(X,Y,e_i,J_\alpha e_i)\\ &\quad+\frac{1}{2n}\{\rho_\beta(J_\gamma Y,X)-\rho_\beta(J_\gamma X,Y)+\rho_\gamma(J_\beta X,Y)-\rho_\gamma(J_\beta Y,X)\}. \end{aligned}\tag{3.39}$$

On the other hand, using (3.30), we calculate

$$\begin{aligned}\sum_{i=1}^{4}D(X,e_i,J_\alpha e_i,Y) &= \sum_{i=1}^{4n}\{R(X,e_i,J_\alpha e_i,Y)+R(Y,e_i,J_\alpha e_i X)\}\\ &= -\mathrm{Ric}(Y,J_\alpha X)-\mathrm{Ric}(X,J_\alpha Y)+\frac{1}{n}\{\rho_\beta(X,J_\gamma Y)\\ &\quad+\rho_\beta(Y,J_\gamma X)-\rho_\gamma(Y,J_\beta X)-\rho_\gamma(X,J_\beta Y)\}.\end{aligned}\tag{3.40}$$

Combining (3.39) and (3.40), we derive

$$\begin{aligned}&n\rho_\alpha(X,J_\alpha Y)+\rho_\beta(X,J_\beta Y)+\rho_\gamma(X,J_\gamma Y)\\ &\quad= -n\,\mathrm{Ric}(XY)+\tfrac{1}{2}nB_\alpha(X,J_\alpha Y)+\tfrac{1}{2}nD_\alpha(X,J_\alpha Y),\end{aligned}\tag{3.41}$$

where the tensors B_α and D_α are defined by $B_\alpha(X,Y)=\sum_{i=1}^{4n}B(X,Y,e_i,J_\alpha e_i)$ and $D_\alpha(X,Y)=\sum_{i=1}^{4n}D(X,e_i,J_\alpha e_i,Y)$. Taking into account (3.38), we get the expression

$$D_\alpha(X,Y)=\frac{1}{2}\sum_{i=1}^{4n}(\nabla_X T)(Y,e_i,J_\alpha e_i)+\frac{1}{2}\sum_{i=1}^{4n}(\nabla_Y T)(X,e_i,J_\alpha e_i),\quad \alpha=1,2,3.\tag{3.42}$$

To calculate $B_\alpha+D_\alpha$, we use (3.35) twice and (3.42). After some calculations, we derive

$$B_\alpha(X,Y)+D_\alpha(X,Y)=\frac{1}{2}\sum_{i=1}^{4n}\mathrm{d}T(X,Y,e_i,J_\alpha e_i)+\sum_{i=1}^{4n}(\nabla_X T)(Y,e_i,J_\alpha e_i),\quad \alpha=1,2,3.\tag{3.43}$$

We substitute (3.43) into (3.41). Solving the obtained system, we obtain

$$\begin{aligned}(n-1)\{\rho_\alpha(X,J_\alpha Y)-\rho_\beta(X,J_\beta Y)\} &= \tfrac{1}{2}n\{(\mathrm{d}T)_\alpha(X,J_\alpha Y)-(\mathrm{d}T)_\beta(X,J_\beta Y)\}\\ &\quad+\tfrac{1}{2}n\{(\nabla T)_\alpha(X,J_\alpha Y)-(\nabla T)_\beta(X,J_\beta Y)\}.\end{aligned}\tag{3.44}$$

Finally, (3.41) and (3.44) imply (3.32). □

Remark 3.1. The Ricci tensor of a QKT connection is not symmetric in general. From (3.34) and (3.36) and the fact that T is a 3-form we get the formula $\mathrm{Ric}(X,Y)-\mathrm{Ric}(Y,X) = \sum_{i=1}^{4n}(\nabla^g_{e_i}T)(e_i,X,Y) = -\delta T(X,Y)$. Hence, the Ricci tensor of a metric linear connection with totally skew-symmetric torsion is symmetric if and only if the torsion 3-form is co-closed.

Corollary 3.4. *Let (M,g,∇,T) be a Riemannian manifold with a metric connection ∇ of totally skew-symmetric torsion T. The following conditions are equivalent*:

1. $\nabla^g T = \frac{1}{4}\,\mathrm{d}T$;
2. ∇T *is a* 4-*form*;
3. $R(X,Y,Z,U) = R(Z,U,X,Y)$.

Proof. Eqs. (3.35) and (3.36) yield

$$\underset{XYZ}{\sigma} R(X,Y,Z,U) - \mathrm{d}T(X,Y,Z,U) + \underset{XYZ}{\sigma}\{g(T(X,Y),T(Z,U))\} = (\nabla_U T)(X,Y,Z).$$

The last equality together with (3.34) and (3.38) lead to the desired equivalencies. □

4. QKT manifolds with parallel torsion and homogeneous QKT structures

Let $(G/K,g)$ be a reductive (locally) homogeneous Riemannian manifold. The canonical connection ∇ is characterised by the properties $\nabla g = \nabla T = \nabla R = 0$ [26, p. 193]. A homogeneous quaternionic Hermitian manifold (resp. homogeneous hyper Hermitian) manifold $(G/K,g,Q)$ is a homogeneous Riemannian manifold with an invariant quaternionic Hermitian subbundle Q (resp. three invariant anti-commuting complex structures). This means that the bundle Q (resp. each of the three complex structures) is parallel with respect to the canonical connection ∇. The torsion of ∇ is totally skew-symmetric if and only if the homogeneous Riemannian manifold is naturally reductive [26] (see also [34,45]). Homogeneous QKT (resp. HKT) manifolds are homogeneous quaternionic Hermitian (resp. homogeneous hyper Hermitian) manifold which are naturally reductive. Examples of homogeneous HKT and QKT manifolds are presented in [34]. The homogeneous QKT manifolds in [34] are constructed from homogeneous HKT manifolds.

In this section, we generalise the result of Opfermann and Papadopoulos [34] which states that there are no homogeneous QKT manifold with torsion 4-form $\mathrm{d}T$ of type $(2,2)$ in dimensions greater than 4. First, we prove the following technical result.

Proposition 4.1. *Let $(M,g,(J_\alpha),\nabla)$ be a $4n$-dimensional $(n>1)$ QKT manifold with 4-form $\mathrm{d}T$ of type $(2,2)$ with respect to each $J_\alpha, \alpha=1,2,3$. Suppose that the torsion is parallel with respect to the QKT connection. Then the Ricci forms ρ_α are given by*

$$\rho_\alpha(X,J_\alpha Z) = \lambda g(X,Y), \quad \alpha=1,2,3, \tag{4.45}$$

where λ is a smooth function on M.

Proof. Let the torsion be parallel, i.e. $\nabla T = 0$. Remark 3.1 shows that the Ricci tensor is symmetric. The equalities (3.35) and (3.36) imply

$$B(X, Y, Z, U) = \underset{XYZ}{\sigma}\{g(T(X, Y), T(Z, U))\} = \tfrac{1}{2}\mathrm{d}T(X, Y, Z, U). \tag{4.46}$$

We get $D = 0$ from Corollary 3.4.

Suppose now that the 4-form $\mathrm{d}T$ is of type $(2, 2)$ with respect to each $J_\alpha, \alpha = 1, 2, 3$. Then it satisfies the equalities

$$\begin{aligned}\mathrm{d}T(X, Y, Z, U) = {}& \mathrm{d}T(J_\alpha X, J_\alpha Y, Z, U) + \mathrm{d}T(J_\alpha X, Y, J_\alpha Z, U)\\ &+ \mathrm{d}T(X, J_\alpha Y, J_\alpha Z, U).\end{aligned} \tag{4.47}$$

The similar arguments as we used in the proof of Proposition 2.1 but applying (4.47) instead of (2.4), yield the following lemma.

Lemma 4.2. *On a QKT manifold with* 4*-form* $\mathrm{d}T$ *of type* $(2, 2)$ *with respect to each* $J_\alpha, \alpha = 1, 2, 3$*, the following equalities hold*:

$$(\mathrm{d}T)_1(X, J_1Y) = (\mathrm{d}T)_2(X, J_2Y) = (\mathrm{d}T)_3(X, J_3Y), \tag{4.48}$$

$$(\mathrm{d}T)_\alpha(X, J_\alpha Y) = -(\mathrm{d}T)_\alpha(J_\alpha X, Y), \quad \alpha = 1, 2, 3. \tag{4.49}$$

We substitute (4.46) and (4.48) and $D = 0$ into (3.34) and (3.44) to get

$$\rho_1(X, J_1Y) = \rho_2(X, J_2Y) = \rho_3(X, J_3Y), \tag{4.50}$$

$$\rho_\alpha(X, J_\alpha Y) = -\frac{n}{n+2}\,\mathrm{Ric}(X, Y) + \frac{n}{4(n+2)}(\mathrm{d}T)_\alpha(X, J_\alpha Y), \quad \alpha = 1, 2, 3. \tag{4.51}$$

The equality (4.49) shows that the 2-form $\mathrm{d}T_\alpha$ is a $(1, 1)$-form with respect to J_α. Hence, the $\mathrm{d}T_\alpha$ is $(1, 1)$-form with respect to each $J_\alpha, \alpha = 1, 2, 3$, because of (4.48). Since the Ricci tensor 'Ric' is symmetric, (4.51) shows that the Ricci tensor Ric is of hybrid type with respect to each J_α, i.e. $\mathrm{Ric}(J_\alpha X, J_\alpha Y) = \mathrm{Ric}(X, Y), \alpha = 1, 2, 3$ and the Ricci forms $\rho_\alpha, \alpha = 1, 2, 3$ are $(1, 1)$-forms with respect to all $J_\alpha, \alpha = 1, 2, 3$. Taking into account (3.30), we obtain

$$\begin{aligned}&R(X, J_\alpha X, Z, J_\alpha Z) + R(X, J_\alpha X, J_\beta Z, J_\gamma Z) + R(J_\beta X, J_\gamma X, Z, J_\alpha Z)\\ &\quad + R(J_\beta X, J_\gamma X, J_\beta Z, J_\gamma Z) = \frac{1}{n}(\rho_\alpha(X, J_\alpha X) + \rho_\alpha(J_\beta X, J_\gamma X))g(Z, Z)\\ &\quad = \frac{2}{n}\rho_\alpha(X, J_\alpha X)g(Z, Z),\end{aligned} \tag{4.52}$$

where the last equality of (4.52) is a consequence of the following identity:

$$\rho_\alpha(J_\beta X, J_\gamma X) = -\rho_\beta(J_\beta X, X) = \rho_\alpha(X, J_\alpha X).$$

The left-hand side of (4.52) is symmetric with respect to the vectors X, Z because $D = 0$. Hence, $\rho_\alpha(X, J_\alpha X)g(Z, Z) = \rho_\alpha(Z, J_\alpha Z)g(X, X)$, $\alpha = 1, 2, 3$. The last equality together with (4.50) implies (4.45). □

Theorem 4.3. *Let* $(M, g, (J_\alpha))$ *be a* $4n$*-dimensional* $(n > 1)$ *QKT manifold with* 4*-form* $\mathrm{d}T$ *of type* $(2, 2)$ *with respect to each* J_α, $\alpha = 1, 2, 3$. *Suppose that the torsion is parallel with respect to the QKT connection. Then* $(M, g, (J_\alpha))$ *is either an HKT manifold with parallel torsion or a QK manifold.*

Proof. We apply Proposition 4.1. If the function $\lambda = 0$ then $\rho_\alpha = 0$, $\alpha = 1, 2, 3$, by (4.45) and Proposition 3.2 implies that the QKT manifold is actually an HKT manifold.

Let $\lambda \neq 0$. The condition (4.45) determines the torsion completely. We proceed involving (3.31) into the computations as in [24]. We calculate using (2.1) and (4.45) that

$$(\nabla_Z \rho_\alpha)(X, Y) = \lambda\{\omega_\beta(Z)F_\gamma(X, Y) - \omega_\gamma(Z)F_\beta(X, Y)\} - \mathrm{d}\lambda(Z)F_\alpha(X, Y). \quad (4.53)$$

Applying the operator d to (3.30), we get taking into account (4.45) that

$$\mathrm{d}\rho_\alpha = \lambda(F_\beta \wedge \omega_\gamma - \omega_\beta \wedge F_\gamma). \quad (4.54)$$

On the other hand, we have

$$\mathrm{d}\rho_\alpha = \underset{XYZ}{\sigma}\{(\nabla_Z \rho_\alpha)(X, Y) + \lambda(T(X, Y), J_\alpha Z)\}, \quad \alpha = 1, 2, 3. \quad (4.55)$$

Comparing the left-hand sides of (4.54) and (4.55) and using (4.53), we derive

$$\lambda \underset{XYZ}{\sigma}\{(T(X, Y), J_\alpha Z)\} = \mathrm{d}\lambda \wedge F_\alpha(X, Y, Z), \quad \alpha = 1, 2, 3.$$

The last equality implies $\lambda T = J_\alpha\, \mathrm{d}\lambda \wedge F_\alpha$, $\alpha = 1, 2, 3$. If λ is a non-zero constant then $T = 0$ and we recover the result of Howe et al. [24]. If λ is not a constant then there exists a point $p \in M$ and a neighbourhood V_p of p such that $\lambda|_{V_p} \neq 0$. Then

$$T = J_\alpha\, \mathrm{d}\ln\lambda \wedge F_\alpha, \quad \alpha = 1, 2, 3. \quad (4.56)$$

We take the trace in (4.56) to obtain

$$4(n-1)J_\alpha\, \mathrm{d}\ln\lambda = 0, \quad \alpha = 1, 2, 3. \quad (4.57)$$

Eq. (4.57) forces $\mathrm{d}\lambda = 0$ since $n > 1$ and, consequently, $T = 0$ by (4.56). Hence, the QKT space is a QK manifold which completes the proof. □

On a locally homogeneous QKT manifold the torsion and curvature are parallel and Theorem 4.3 leads to the following.

Theorem 4.4. *A* (*locally*) *homogeneous* $4n$*-dimensional* $(n > 1)$ *QKT manifold with torsion* 4*-form* $\mathrm{d}T$ *of type* $(2, 2)$ *is either* (*locally*) *homogeneous HKT space or a* (*locally*) *symmetric QK space.*

Theorem 4.4 shows that there are no homogeneous (proper) QKT manifolds with torsion 4-form of type $(2, 2)$ in dimensions greater than 4 which is proved in [34] by different methods using the Lie algebra arguments.

5. Four-dimensional QKT manifolds

In dimension 4, the situation is completely different from that described in Theorems 2.2 and 4.3 in higher dimensions. For a given quaternionic structure on a four-dimensional manifold $(M, g(H))$ (or equivalently, given an orientation and a conformal class of Riemannian metrics [19]) there are many QKT structures [24]. More precisely, all QKT structures associated with $(g, (H))$ depend on a 1-form ψ due to the general identity

$$*\psi = -J\psi \wedge F, \tag{5.58}$$

where $*$ is the Hodge $*$-operator, J is a g-orthogonal almost complex structure with Kähler form F (see [19]). Indeed, for any given 1-form ψ, we may define a QKT connection ∇ as follows: $\nabla = \nabla^g + \frac{1}{2} * \psi$. Conversely, any 3-form T can be represented by $T = -*(*T)$ and the connection given above is a quaternionic connection with torsion $T = *\psi$. Hence, a QKT structure on a four-dimensional oriented manifold is a pair (g, t) of a Riemannian metric g and a 1-form t. The choice of g generates three almost complex structures (J_α), $\alpha = 1, 2, 3$, satisfying the quaternionic identities [19]. The torsion 3-form T is given by

$$T = *t = t_\alpha \wedge F_\alpha = t_\beta \wedge F_\beta = t_\gamma \wedge F_\gamma. \tag{5.59}$$

As a consequence of (5.58), we obtain $*\mathrm{d}T = *\mathrm{d} * t = -\delta t$. The last identity means that the torsion 3-form T is closed if and only if the 1-form t is co-closed. Thus, in dimension 4, there are many strong QKT structures.

In higher dimensions the conformal change of the metric induces a unique QKT structure by Theorem 2.7. We may define a QKT connection corresponding to a conformally equivalent metric $\bar{g} = fg$ in dimension 4 by (2.24) and call this conformal QKT transformation. In the compact case, taking the Gauduchon metric of Theorem 2.7, we obtain the following proposition.

Proposition 5.1. *Let $(M, g, (H), \nabla)$ be a compact four-dimensional QKT manifold. In the conformal class $[g]$ there exists a unique (up to homotety) strong QKT structure conformally equivalent to the given one.*

Further, we consider QKT structures with parallel torsion. We have the following theorem.

Theorem 5.2. *A four-dimensional QKT manifold M with parallel torsion 3-form is a strong QKT manifold, the torsion 1-form is parallel with respect to the Levi-Civita connection and M is locally isometric to the product $N^3 \times \mathcal{R}$, where N^3 is a three-dimensional Riemannian manifold admitting a Riemannian connection ∇ with totally skew-symmetric torsion, parallel with respect to ∇.*

Proof. The proof is based on the following lemma.

Lemma 5.3. *A four-dimensional QKT manifold has parallel torsion* 3*-form if and only if it has parallel torsion* 1*-form with respect to the Levi-Civita connection.*

Proof. We calculate using (2.1) and (5.59) that

$$\begin{aligned}(\nabla_Z T)(X, Y, U) = {} & t_\alpha(U)(\omega_\beta(Z)F_\gamma(Y, X) - \omega_\gamma(Z)F_\beta(Y, X)) \\ & - t_\alpha(X)(\omega_\beta(Z)F_\gamma(Y, U) - \omega_\gamma(Z)F_\beta(Y, U)) \\ & + t_\alpha(Y)(\omega_\beta(Z)F_\gamma(X, U) - \omega_\gamma(Z)F_\beta(X, U)) \\ & + F_\alpha(Y, U)(\nabla_Z t_\alpha)X + F_\alpha(X, Y)(\nabla_Z t_\alpha)U - F_\alpha(X, U)(\nabla_Z t_\alpha)Y.\end{aligned} \tag{5.60}$$

Taking the trace in (5.60), we obtain

$$\sum_{i=1}^{4}(\nabla_Z T)(X, e_i, J_\alpha e_i) = -2(\nabla_Z t_\alpha)X - 2(\omega_\beta(Z)t_\gamma(X) - \omega_\gamma(Z)t_\beta(X)). \tag{5.61}$$

Using (2.1), we get

$$(\nabla_Z t_\alpha)X = (\nabla_Z t)J_\alpha X - (\omega_\beta(Z)t_\gamma(X) - \omega_\gamma(Z)t_\beta(X)). \tag{5.62}$$

Eqs. (5.61) and (5.62) yield

$$\sum_{i=1}^{4}(\nabla_Z T)(J_\alpha X, e_i, J_\alpha e_i) = 2(\nabla_Z t)X, \quad \alpha = 1, 2, 3. \tag{5.63}$$

Then $\nabla t = 0$, since the torsion is parallel. But $\nabla^g t = \nabla t$ by (2.11) and (5.59). Hence, $\nabla^g t = 0$.

For the converse, we insert (5.62) into (5.60) to get

$$\begin{aligned}(\nabla_Z T)(X, Y, U) = {} & F_\alpha(Y, U)(\nabla_Z t)J_\alpha X + F_\alpha(X, Y)(\nabla_Z t)J_\alpha U \\ & + F_\alpha(U, X)(\nabla_Z t)J_\alpha Y,\end{aligned} \tag{5.64}$$

since the dimension is equal to 4. If $\nabla^g t = 0$ then $\nabla t = 0$ and (5.64) leads to $\nabla T = 0$ which proves the lemma. □

Lemma 5.3 shows that (M, g) is locally isometric to the Riemannian product $\mathcal{R} \times N^3$ of a real line and a three-dimensional manifold N^3 (see, e.g. [26]). Using (5.59), we see that $T(t^\#, X^\perp, Y^\perp) = 0$ for every vector fields $X^\perp, Y^\perp$ orthonormal to the vector field $t^\#$ dual to the torsion 1-form t. Hence, the torsion T and therefore the connection ∇ descend to N^3.

In particular, $\delta t = 0$ and therefore the QKT structure is strong. □

As a consequence of Theorem 5.2, we recover the following two results proved in [27] in the setting of naturally reductive homogeneous 4-manifolds.

Theorem 5.4. *A* (*locally*) *homogeneous four-dimensional QKT manifold is locally isometric to the Riemannian product* $\mathcal{R} \times N^3$ *of a real line and a naturally reductive homogeneous* 3*-manifold* N^3.

Theorem 5.5. *Let (M, g) be a four-dimensional compact homogeneous QKT manifold. Then the universal covering space $\tilde{M}$ of M is isometric to the Riemannian product $\mathcal{R} \times N^3$ of a real line and the three-dimensional space N^3 is one of the following*:

1. $R^3, S^3, \mathcal{H}^3$.
2. *Isometric to one of the following Lie groups with a suitable left invariant metric:*
   2.1. SU(2);
   2.2. $\tilde{\mathrm{SL}(2, \mathcal{R})}$, *the universal covering of* $\mathrm{SL}(2, \mathcal{R})$;
   2.3. *the Heisenberg group.*

Theorem 5.5 is based on the classification of three-dimensional simply connected naturally reductive homogeneous spaces given in [45].

5.1. Einstein-like QKT 4-manifolds

It is well known [1,7] that a $4n$-dimensional $(n > 1)$ QK manifold is Einstein and the Ricci forms satisfy $\rho_\alpha(X, J_\alpha Y) = \rho_\beta(X, J_\beta Y) = \rho_\gamma(X, J_\gamma Y) = \lambda g(X, Y)$, where λ is a constant. However, the assumptions that these properties hold on a QKT manifold $(n > 1)$ force the torsion to be zero [24] and the QKT manifold is a QK manifold. Actually, we have already generalised this result proving that if λ is not a constant the torsion has to be zero (see the proof of Theorem 4.3).

If the dimension is equal to 4, the situation is different. In this section, we show that there exists a four-dimensional (proper) QKT manifold satisfying similar curvature properties as those mentioned above.

We denote by K the following $(0, 2)$ tensor:

$$K(X, Y) := \rho_\alpha(X, J_\alpha Y) + \rho_\beta(X, J_\beta Y) + \rho_\gamma(X, J_\gamma Y).$$

The tensor K is independent of the chosen local almost complex structures (J_α) because of the following proposition.

Proposition 5.6. *Let $(M, g, (J_\alpha), \nabla)$ be a four-dimensional QKT manifold. Then*:

$$K = -\mathrm{Ric} + \nabla^g t - \tfrac{1}{2}(\delta t) g, \tag{5.65}$$

$$\mathrm{Skew}(\mathrm{Ric}) = -\tfrac{1}{4}\langle \mathrm{d}t, F_\alpha\rangle F_\alpha - \tfrac{1}{2} J_\alpha(\mathrm{d}t'), \quad \alpha = 1, 2, 3, \tag{5.66}$$

$$\mathrm{Ric}^g = \mathrm{Sym}(\mathrm{Ric}) + \tfrac{1}{2}(|t|^2 g - t \otimes t), \tag{5.67}$$

where $\langle , \rangle$ is the scalar product of tensors induced by g, Skew (*resp.* Sym) *denotes the skew-symmetric* (*resp. symmetric*) *part of a tensor.*

In particular, the Ricci tensor is symmetric if and only if the torsion 1*-form is closed.*

Proof. We use (3.41). From (3.42) and (5.63), we obtain

$$D_\alpha(X, J_\alpha Y) = (\nabla_X t)Y - (\nabla_{J_\alpha Y} t) J_\alpha X, \quad \alpha = 1, 2, 3. \tag{5.68}$$

To compute B_α, we need the following general identity.

Lemma 5.7. *On a four-dimensional QKT manifold, we have* $\underset{XYZ}{\overset{\sigma}{}} g(T(X,Y),T(Z,U)) = 0$.

Proof. Since $\underset{XYZ}{\overset{\sigma}{}} g(T(X,Y),T(Z,U))$ is a 4-form, it is sufficient to check the equality for a basis of type $\{X, J_\alpha X, J_\beta X, J_\gamma X\}$. The last claim is obvious because of (5.59).

For each $\alpha \in \{1, 2, 3\}$, Lemma 5.7, (5.63) and (5.64) yield

$$B_\alpha(X, J_\alpha Y) = \sum_{i=1}^{4} \underset{XJ_\alpha Ye_i}{\overset{\sigma}{}} (\nabla_X T)(J_\alpha Y, e_i, J_\alpha e_i) = (\nabla_X t)Y + (\nabla_{J_\alpha Y} t)J_\alpha X - \delta t g(X,Y). \tag{5.69}$$

Substituting (5.68) and (5.69) into (3.41) and putting $n = 1$, we derive (5.65) since $\nabla^g t = \nabla t$. Taking the trace in (5.64), we get $\sum_{i=1}^{4}(\nabla_{e_i}T)(e_i, X, Y) = \frac{1}{2}\sum_{i=1}^{4} dt(e_i, J_\alpha e_i)F_\alpha(X,Y) + dt(J_\alpha X, J_\alpha Y)$, $\alpha = 1, 2, 3$. Then (5.67) follows from the last equality and Remark 3.1. Eq. (5.67) is a direct consequence of (3.37) and (5.59). □

A $4n$-dimensional QKT manifold $(M, g, (J_\alpha), \nabla)$ is said to be a *Einstein QKT manifold* if the symmetric part Sym(Ric) of the Ricci tensor of ∇ is a scalar multiple of the metric g, i.e. $\mathrm{Sym(Ric)} = (\mathrm{Scal}/4n)g$, where $\mathrm{Scal} = \mathrm{tr}_g \,\mathrm{Ric}$ is the scalar curvature of ∇.

We note that the scalar curvature 'Scal' of a Einstein QKT manifold may not be a constant.

We shall say that a four-dimensional QKT manifold is *SP(1)-Einstein* if the symmetric part $\mathrm{Sym}(K)$ of the tensor K is a scalar multiple of the metric g since the tensor K is determined by the SP(1)-part of the curvature. On an SP(1)-Einstein QKT manifold $\mathrm{Sym}(K) = \frac{1}{4}(\mathrm{Scal}^K)g$, where $\mathrm{Scal}^K = \mathrm{tr}_g K$.

For a given QKT manifold with torsion 1-form t, we consider the corresponding Weyl structure ∇^{W}, i.e. the unique torsion-free linear connection determined by the condition

$$\nabla^{\mathrm{W}} g = -t \otimes g. \tag{5.70}$$

Conversely, in dimension 4, to a given Weyl structure $\nabla^{\mathrm{W}} g = \psi \otimes g$, we associate the QKT connection with torsion $T = *(-\psi)$. Note that a given Weyl structure on a conformal manifold $(M, [g])$ does not depend on the particularly chosen metric $g \in [g]$, but depends on the conformal class $[g]$. A Weyl structure is said to be *Einstein–Weyl* if the symmetric part $\mathrm{Sym}(\mathrm{Ric}^{\mathrm{W}})$ of its Ricci tensor is a scalar multiple of the metric g. Weyl structures and especially Einstein–Weyl structures have been much studied. For a nice overview of Einstein–Weyl geometry, see [13]. The next theorem shows the link between Einstein–Weyl geometry and SP(1)-Einstein QKT manifolds in dimension 4.

Theorem 5.8. *Let* $(M, g, (J_\alpha), \nabla)$ *be a four-dimensional QKT manifold with torsion* 1*-form* t. *The following conditions are equivalent*:

1. $(M, g, (J_\alpha), \nabla)$ *is an* SP(1)*-Einstein QKT manifold.*
2. *The corresponding Weyl structure is a Einstein–Weyl structure.*

Proof. The Weyl connection ∇^{W} determined by (5.70) is given explicitly by

$$\nabla^{\mathrm{W}}_X Y = \nabla^g_X Y + \tfrac{1}{2}t(X)Y + \tfrac{1}{2}t(Y)X - \tfrac{1}{2}g(X,Y)t^{\#}.$$

The symmetric part of its Ricci tensor is equal to

$$\mathrm{Sym}(\mathrm{Ric}^{\mathrm{W}}) = \mathrm{Ric}^g - \mathrm{Sym}(\nabla^g t) - \tfrac{1}{2}(|t|^2 g - t\otimes t) + \tfrac{1}{2}(\delta t)g. \tag{5.71}$$

Keeping in mind that $\nabla^g t = \nabla t$, we get from (5.65), (5.67) and (5.71) that $\mathrm{Sym}(\mathrm{Ric}^{\mathrm{W}}) = -\mathrm{Sym}(K)$. The theorem follows from the last equality. □

It is well known [6,43] that if there exists a hypercomplex structure on a four-dimensional conformal manifold then the conformal structure has anti-self-dual Weyl tensor (see also [19]). Every four-dimensional hypercomplex manifold $(M, g, (H_\alpha))$, i.e. (an oriented anti-self-dual 4-manifold) carries a unique HKT structure in view of the results in [17,19]. Indeed, let $\theta = \theta_\alpha = \theta_\beta = \theta_\gamma$ be the common Lie form. The unique HKT structure is defined by $\nabla = \nabla^g - \frac{1}{2} * \theta$ [19] (the uniqueness is a consequence of a general result in [17], see also [20]). The HKT structure on a four-dimensional hypercomplex manifold is SP(1)-Einstein since the tensor K vanishes. The corresponding Weyl structure to the given HKT structure on a four-dimensional hyper Hermitian manifold is the Obata connection [19], i.e. the unique torsion-free linear connection which preserves each of the three hypercomplex structures. As a consequence of Theorem 5.8, we recover the result in [39] which states that the Obata connection of a hypercomplex 4-manifold is Einstein–Weyl and the symmetric part of its Ricci tensor is zero.

Theorem 5.8 and (5.65) show that every Einstein–Weyl structure determined by (5.70) on a four-dimensional conformal manifold whose vector field dual to the 1-form t is Killing, induces a Einstein and SP(1)-Einstein QKT structure.

Corollary 5.9. *Let $(M, [g], \nabla^{\mathrm{W}})$ be a compact four-dimensional Einstein–Weyl manifold. Then the corresponding QKT structure to the Gauduchon metric of ∇^{W} is Einstein and* SP(1)-*Einstein.*

Proof. On a compact Einstein–Weyl manifold the vector field dual to the Lie form of the Gauduchon metric is Killing by the result of Tod [44]. Then the claim follows from Theorem 5.8 and (5.65). □

The Ricci tensor of a four-dimensional QKT manifold is symmetric iff the torsion 1-form is closed by Proposition 5.6. Applying Theorem 3 in [16] and using Theorem 5.8, we obtain the following corollary.

Corollary 5.10. *Let $(M, g, (J_\alpha), \nabla)$ be a four-dimensional compact* SP(1)-*Einstein QKT manifold with symmetric Ricci tensor. Suppose that the torsion* 1-*form is not exact. Then the torsion* 1-*form corresponding to the Gauduchon metric g_{G} of $(M, g, (J_\alpha), \nabla)$ is parallel with respect to the Levi-Civita connection of g_{G} and the universal cover of (M, g_{G}) is isometric to $\mathcal{R} \times S^3$. In particular, the quaternionic bundle (J_α) admits hypercomplex structure.*

A lot is known about Einstein–Weyl manifolds (see a nice survey [13]). There are many (compact) Einstein–Weyl 4-manifolds (e.g. $S^2 \otimes S^2$). Among them there are (anti)-self-dual as well as non-(anti)-self-dual. We mention here the Einstein–Weyl examples of Bianchi IX type metric [11,28–30]. All these Einstein–Weyl 4-manifolds admit SP(1)-Einstein QKT structures by Theorem 5.8.

It is also known that there are obstructions to the existence of Einstein–Weyl structures on compact 4-manifold [38]. If the manifold M is finitely covered by $T^2 \otimes S^2$ which cannot be Einstein–Weyl then M does not admit Einstein–Weyl structure and therefore there are no SP(1)-Einstein structures on M.

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